Reporting assignment

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Solve the following problems, and submit the report to me by e-mail.

The operator norm $\|\cdot\|_\infty$, the trace norm $\|\cdot\|_{\mathrm{Tr}}$ and the Frobenius norm $\|\cdot\|_F$ are defined as in the lecture:

$$\|A\|_\infty = \sigma_1(A), \quad \|A\|_{\mathrm{Tr}} = \sum_{i=1}^{q} \sigma_i(A), \qquad \|A\|_F = \sqrt{\sum_{i=1}^{q} \sigma_i^2(A)},$$

for $A \in \mathbb{R}^{M\times N}$ where $\sigma_i(A)$ is the i-th largest singular value, and $q = \min\{M, N\}$.

1. Consider a linear model
$$Y = \mathbf{X}\beta^* + \epsilon$$
where $Y \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{R}^{n\times p}$, and $\epsilon = (\epsilon_i)_{i=1}^n \in \mathbb{R}^n$. Suppose that ϵ_i is an i.i.d. noise such that $\mathrm{E}[\epsilon_i] = 0$ and $\mathrm{E}[\epsilon_i^2] = \sigma^2$, and $\mathbf{X}^\top\mathbf{X}$ is invertible. In this setting, evaluate the in-sample predictive error
$$\mathrm{E}_{Y|X}\left[\frac{1}{n}\|\mathbf{X}\beta^* - \mathbf{X}\hat{\beta}_{\mathrm{LS}}\|^2\right].$$
of the least squared estimator $\hat{\beta}_{\mathrm{LS}} \in \mathbb{R}^p$ by using n, p, σ^2.
2. Prove that, for given $q \in \mathbb{R}$, it holds that
$$\arg\min_{x\in\mathbb{R}} \left\{\frac{1}{2}(x-q)^2 + C|x|\right\} = (|q| - C)_+ \mathrm{sign}(q).$$
3. Prove that
$$\|A\|_{\mathrm{Tr}}/\sqrt{d} \le \|A\|_F \le \|A\|_{\mathrm{Tr}}$$
for all matrix $A \in \mathbb{R}^{M\times N}$ that has rank d.
4. Let $U = [U_1|U_2] \in O(M)$ and $V = [V_1|V_2] \in O(N)$ where $U_1 \in \mathbb{R}^{M\times d}$ and $V_1 \in \mathbb{R}^{N\times d}$. For a matrix $A \in \mathbb{R}^{M\times N}$, let $A_1 = A - (U_2U_2^\top)A(V_2V_2^\top)$. Prove that the rank of A_1 is at most $2d$.
5. Prove that the i.i.d. standard normal random variable sequence $g_i \sim N(0,1)$ $(i = 1,\dots,n)$ satisfies
$$\frac{\sqrt{n}}{2} \le \mathrm{E}\left[\sqrt{\sum\nolimits_{i=1}^n g_i^2}\right] \le \sqrt{n}.$$
6. Write your idea about an application of low rank matrix estimation that you think is interesting.
7. (optional) Apply a low rank decomposition technique to some problems and report its result.