Stochastic Optimization
Part III: Advanced topics of stochastic optimization

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Outline

1. Stochastic optimization for structured regularization
   - Structured regularization
   - Alternating Direction Method of Multipliers (ADMM)
   - Stochastic ADMM for online data
   - Stochastic ADMM for batch data

2. Parallel and distributed optimization

3. Further interesting topics
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1. Stochastic optimization for structured regularization
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Regularized learning problem

Lasso:

\[ \min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} (y_i - z_i^\top x)^2 + \|x\|_1. \]
Regularized learning problem

Lasso:

$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} (y_i - z_i^T x)^2 + \|x\|_1$$

General regularized learning problem:

$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(z_i^T x) + \tilde{\psi}(x).$$

**Difficulty:** Sparsity inducing regularization is usually **non-smooth**.
**Proximal mapping**

Regularized learning problem:

\[
\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} f_i(z_i^T x) + \tilde{\psi}(x).
\]

\[
g_t \in \partial_x \left( \frac{1}{n} \sum_{i=1}^{n} f_i(z_i^T x) \right) \big|_{x=x(t-1)}, \quad \bar{g}_t = \frac{1}{t} \sum_{\tau=1}^{t} g_\tau.
\]

Proximal gradient descent:

\[
x^{(t)} = \arg \min_{x \in \mathbb{R}^p} \left\{ g_t^T x + \tilde{\psi}(x) + \frac{1}{2\eta_t} \| x - x^{(t-1)} \|^2 \right\}.
\]

Regularized dual averaging:

\[
x^{(t)} = \arg \min_{x \in \mathbb{R}^p} \left\{ \bar{g}_t^T x + \tilde{\psi}(x) + \frac{1}{2\eta_t} \| x \|^2 \right\}.
\]

A key computation is the **proximal mapping**:

\[
\text{prox}(q | \tilde{\psi}) := \arg \min_{x} \left\{ \tilde{\psi}(x) + \frac{1}{2} \| x - q \|^2 \right\}.
\]
Example of Proximal mapping: $\ell_1$ regularization

$$\text{prox}(q \mid C \| \cdot \|_1) = \arg \min_x \left\{ C\|x\|_1 + \frac{1}{2}\|x - q\|^2 \right\}$$

$$= (\text{sign}(q_j) \max(|q_j| - C, 0))_j.$$  

→ Soft-thresholding function. **Analytic form.**

There are also many regularization functions for which computing the prox mapping is difficult.  
→ **Structured regularization.**
Examples of structured regularization

- Overlapped group lasso

\[ \tilde{\psi}(w) = C \sum_{g \in G} \| w_g \|_q \]

\((q > 1; \text{typically } q = 2, \infty)\)

- The groups may have overlap.
- It is difficult to compute the proximal mapping.

Application (1)

Genome Wide Association Study (GWAS)
(Balding ‘06, McCarthy et al. ‘08)
Sentence regularization for text classification (Yogatama and Smith, 2014)

The words occurred in the same sentence is grouped:

\[
\hat{\psi}(w) = \sum_{d=1}^{D} \sum_{s=1}^{S_d} \lambda_{d,s} \| w(d,s) \|_2,
\]

\((d \text{ expresses a document, } s \text{ expresses a sentence}).\)

<table>
<thead>
<tr>
<th>Sentence</th>
<th>Negative</th>
<th>Positive</th>
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</thead>
<tbody>
<tr>
<td>this film is one big joke : you have all the basics elements</td>
<td></td>
<td></td>
</tr>
<tr>
<td>of romance (love at first sight, great passion, etc.) and gangster flicks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(brutality, dagerous machinations, the mysterious don, etc.),</td>
<td></td>
<td></td>
</tr>
<tr>
<td>but it is all done with the crudest humor.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>it’s the kind of thing you either like viserally and</td>
<td></td>
<td></td>
</tr>
<tr>
<td>immediately ” get ” or you don’t.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>that is a matter of taste and expectations.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>i enjoyed it and it took me back to the mid80s,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>when nicolson and turner were in their primes.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>the acting is very good, if a bit obviously tongue - in - cheek.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(Generalized) Fused Lasso and TV-denoising

Fused lasso (Tibshirani et al. (2005), Jacob et al. (2009)):

\[
\psi(x) = C \sum_{(i,j) \in E} |x_i - x_j|.
\]

TV-denoising (Rudin et al., 1992, Chambolle, 2004):

\[
\psi(X) = C \sum_{i,j} \sqrt{|X_{i+1,j} - X_{i,j}|^2 + |X_{i,j+1} - X_{i,j}|^2}.
\]

Genome data analysis by Fused lasso (Tibshirani and Taylor, 2011)

Image restoration (Mairal et al., 2009)
Other examples

- Robust PCA (Candés et al. 2009).
- Low rank tensor estimation (Signoretto et al., 2010; Tomioka et al., 2011).
- Dictionary learning (Kasiviswanathan et al., 2012; Rakotomamonjy, 2013).
Solutions

- Developing a sophisticated method for each regularization: Jacob et al. (2009), Yuan et al. (2011).
- Submodular optimization: Bach (2010), Kawahara et al. (2009), Bach et al. (2012)
- Decomposing the proximal mapping: Yu (2013)
- Applying linear transformation to make the regularization simpler.
  → Alternating Direction Method of Multipliers, ADMM.
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Linear transformation, Decomposition technique

- Overlapped group lasso $\tilde{\psi}(w) = C \sum_{g \in \mathcal{G}} \| w_g \|

※ It is difficult to compute the proximal mapping.

**Solution:**

- Prepare $\psi$ for which proximal mapping is easily computable.
- Let $\psi(B^\top w) = \tilde{\psi}(w)$, and utilize the proximal mapping w.r.t. $\tilde{\psi}$.

Decompose into independent groups:

$$B^\top w = \begin{bmatrix} w_{g_1} \\ w_{g_2} \\ w_{g_3} \end{bmatrix}$$

$$\psi(y) = C \sum_{g' \in \mathcal{G}'} \| y_{g'} \|$$

$$\text{prox}(q|\psi) = \left( q_{g'} \max \left\{ 1 - \frac{C}{\| q_{g'} \|}, 0 \right\} \right)_{g' \in \mathcal{G}'}$$
Another example

- **Graph guided regularization**

\[ \tilde{\psi}(w) = C \sum_{(i,j) \in E} |w_i - w_j|. \]

\[ \psi(y) = C \sum_{e \in E} |y_e|, \quad y = B^\top w = (w_i - w_j)_{(i,j) \in E} \]

\[ \Rightarrow \begin{cases} 
\psi(B^\top w) = \tilde{\psi}(w), \\
\text{prox}(q|\psi) = \left(q_e \max \left\{1 - \frac{c}{|q_e|}, 0\right\}\right)_{e \in E}. 
\end{cases} \]

Soft-Thresholding function.
Optimizing composite objective function with linear constraint

\[
\min_x \frac{1}{n} \sum_{i=1}^{n} f_i(z_i^T x) + \psi(B^T x)
\]

\[
\Leftrightarrow \min_{x,y} \frac{1}{n} \sum_{i=1}^{n} f_i(z_i^T x) + \psi(y) \quad \text{s.t.} \quad y = B^T x.
\]

Augmented Lagrangian

\[
\mathcal{L}(x, y, \lambda) = \frac{1}{n} \sum_{i} f_i(z_i^T x) + \psi(y) + \lambda^T (y - B^T x) + \frac{\rho}{2} \|y - B^T x\|^2
\]

\[
\inf_{x,y} \sup_{\lambda} \mathcal{L}(x, y, \lambda)
\]

yields the optimization of the original problem.

Method of multipliers

\[
\min_{x,y} \{ f(x) + \psi(y) \quad \text{s.t.} \quad Ax + By = 0 \}
\]

\[
\mathcal{L}(w, y, \lambda) = f(x) + \psi(y) + \lambda^\top (Ax + By) + \frac{\rho}{2} \|Ax + By\|^2
\]

Method of multipliers (Hestenes, 1969, Powell, 1969)

\[
(x^{(t)}, y^{(t)}) = \arg\min_{(x,y)} \mathcal{L}(x, y, \lambda^{(t-1)})
\]

\[
\lambda^{(t)} = \lambda^{(t-1)} + \rho(Ax^{(t)} + By^{(t)})
\]
Method of multipliers

\[
\min_{x,y} \{ f(x) + \psi(y) \} \quad \text{s.t.} \quad Ax + By = 0
\]

\[
\mathcal{L}(w, y, \lambda) = f(x) + \psi(y) + \lambda^\top (Ax + By) + \frac{\rho}{2} \|Ax + By\|^2
\]

Method of multipliers (Hestenes, 1969, Powell, 1969)

\[
(x^{(t)}, y^{(t)}) = \text{argmin}_{x,y} \mathcal{L}(x, y, \lambda^{(t-1)})
\]

\[
\lambda^{(t)} = \lambda^{(t-1)} + \rho (Ax^{(t)} + By^{(t)})
\]

Remark:

- Update of \( \lambda \) corresponds to gradient ascent of \( \mathcal{L} \).
- It is easy to check that

\[
\nabla_x (f(x) + \langle \lambda^{(t)}, Ax + By^{(t)} \rangle)\big|_{x=x^{(t)}} = 0
\]

\[
\nabla_y (\psi(y) + \langle \lambda^{(t)}, Ax^{(t)} + By \rangle)\big|_{y=y^{(t)}} = 0.
\]

If \( Ax^{(t)} + By^{(t)} = 0 \), this gives the optimality condition.
Alternating Direction Method of multipliers
(Douglas-Rachford splitting)

\[
\min_{x, y} \left\{ f(x) + \psi(y) \right\} \text{ s.t. } Ax + By = 0
\]

\[
\mathcal{L}(w, y, \lambda) = f(x) + \psi(y) + \lambda^\top (Ax + By) + \frac{\rho}{2} \|Ax + By\|^2
\]

Alternating Direction Method of Multipliers (Gabay and Mercier, 1976)

\[
x^{(t)} = \arg\min_x \mathcal{L}(x, y^{(t-1)}, \lambda^{(t-1)})
\]

\[
y^{(t)} = \arg\min_y \mathcal{L}(x^{(t)}, y, \lambda^{(t-1)})
\]

\[
\lambda^{(t)} = \lambda^{(t-1)} + \rho (Ax^{(t)} + By^{(t)})
\]
Alternating Direction Method of multipliers
(Douglas-Rachford splitting)

$$\min_{x,y} \left\{ \ f(x) + \psi(y) \ \text{s.t.} \ Ax + By = 0 \right\}$$

$$L(w, y, \lambda) = f(x) + \psi(y) + \lambda^T (Ax + By) + \frac{\rho}{2} \|Ax + By\|^2$$

Alternating Direction Method of Multipliers (Gabay and Mercier, 1976)

$$x^{(t)} = \arg \min_x L(x, y^{(t-1)}, \lambda^{(t-1)})$$

$$y^{(t)} = \arg \min_y L(x^{(t)}, y, \lambda^{(t-1)})$$

$$\lambda^{(t)} = \lambda^{(t-1)} + \rho(Ax^{(t)} + By^{(t)})$$

- ADMM converges to the optimal (Mota et al., 2011)
- $O(1/t)$ convergence in general (He and Yuan, 2012)
- Linear convergence for a strongly convex objective (Deng and Yin, 2012, Hong and Luo, 2012)
ADMM for structured regularization

\[
\min_x \{ f(x) + \psi(B^T w) \} \Leftrightarrow \min_{x, y} \{ f(x) + \psi(y) \text{ s.t. } y = B^T x \}
\]

\[
\mathcal{L}(x, y, \lambda) = f(x) + \psi(y) + \lambda^T (y - B^T x) + \frac{\rho}{2} \| y - B^T x \|^2
\]

where \( f(x) = \frac{1}{n} \sum f_i(z_i^T x) \)

**ADMM for structured regularization**

\[
x^{(t)} = \arg \min_x \{ f(x) + \lambda^{(t-1)^T} (-B^T x) + \frac{\rho}{2} \| y^{(t-1)} - B^T x \|^2 \}
\]

\[
y^{(t)} = \arg \min_y \{ \psi(y) + \lambda^{(t)^T} y + \frac{\rho}{2} \| y - B^T x^{(t)} \|^2 \}
\]

\[
(= \text{prox}(B^T x^{(t)} - \lambda^{(t)}/\rho |\psi/\rho))
\]

\[
\lambda^{(t)} = \lambda^{(t-1)} - \rho (B^T x^{(t)} - y^{(t)})
\]

- The update of \( y \) is given by the proximal mapping \( \text{w.r.t. simple } \psi \).
  → Usually analytic form.
**ADMM for structured regularization**

\[
\min_{x} \{ f(x) + \psi(B^\top w) \} \iff \min_{x,y} \{ f(x) + \psi(y) \text{ s.t. } y = B^\top x \}
\]

\[
\mathcal{L}(x, y, \lambda) = f(x) + \psi(y) + \lambda^\top (y - B^\top x) + \frac{\rho}{2} \|y - B^\top x\|_2^2
\]

where \( f(x) = \frac{1}{n} \sum f_i(z_i^\top x) \)

---

**ADMM for structured regularization**

\[
x(t) = \arg \min_{x} \{ f(x) + \lambda^{(t-1)^\top} (-B^\top x) + \frac{\rho}{2} \|y^{(t-1)} - B^\top x\|_2^2 \}
\]

\[
y(t) = \arg \min_{y} \{ \psi(y) + \lambda^{(t)}^\top y + \frac{\rho}{2} \|y - B^\top x^{(t)}\|_2^2 \}
\]

\[
(= \text{prox}(B^\top x^{(t)} - \lambda^{(t)}/\rho |\psi/\rho))
\]

\[
\lambda^{(t)} = \lambda^{(t-1)} - \rho(B^\top x^{(t)} - y^{(t)}).
\]

- The update of \( y \) is given by the proximal mapping w.r.t. simple \( \psi \).
  → Usually analytic form.

- However, the computation of \( \frac{1}{n} \sum_{i=1}^{n} f_i(z_i^\top w) \) is still heavy.
  → **Stochastic version** of ADMM has been developed (Suzuki, 2013, Ouyang et al., 2013, Suzuki, 2014).
Example: ADMM for Lasso

\[
\min_{x} \left\{ \frac{1}{2n} \| ZX - Y \|^2 + C \| x \|_1 \right\}
\]

**ADMM for Lasso**

\[
x^{(t)} = \left( \frac{Z^T Z}{2n} + \frac{\rho I}{2} \right)^{-1} \left( \frac{Y}{n} + \lambda^{(t)} - \rho y^{(t-1)} \right)
\]

\[
y^{(t)} = ST_{\frac{C}{\rho}}(x^{(t)} - \lambda^{(t)}/\rho)
\]

\[
\lambda^{(t)} = \lambda^{(t-1)} - \rho(x^{(t)} - y^{(t)})
\]

\[
ST_{\eta}(x) = (\text{sign}(x_j) \max\{|x_j| - \eta, 0\})_j
\]
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## Table of literature

Stochastic methods for regularized learning problems.

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<th></th>
<th>Normal</th>
<th>ADMM</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Online</strong></td>
<td>• Proximal gradient type (Nesterov, 2007)</td>
<td>Online-ADMM (Wang and Banerjee, 2012)</td>
</tr>
<tr>
<td></td>
<td>FOBOS (Duchi and Singer, 2009)</td>
<td>SGD-ADMM (Suzuki, 2013, Ouyang et al., 2013)</td>
</tr>
<tr>
<td></td>
<td>• Dual averaging type (Nesterov, 2009)</td>
<td>RDA-ADMM (Suzuki, 2013)</td>
</tr>
<tr>
<td></td>
<td>RDA (Xiao, 2009)</td>
<td></td>
</tr>
<tr>
<td><strong>Batch</strong></td>
<td>SDCA (Shalev-Shwartz and Zhang, 2013)</td>
<td>SDCA-ADMM (Suzuki, 2014)</td>
</tr>
<tr>
<td></td>
<td>(Stochastic Dual Coordinate Ascent)</td>
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<tr>
<td></td>
<td>SAG (Le Roux et al., 2013)</td>
<td></td>
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<tr>
<td></td>
<td>(Stochastic Averaging Gradient)</td>
<td></td>
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<tr>
<td></td>
<td>SVRG (Johnson and Zhang, 2013)</td>
<td></td>
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<tr>
<td></td>
<td>(Stochastic Variance Reduced Gradient)</td>
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</tbody>
</table>

**Online type:** $O(1/\sqrt{T})$ in general, $O(\log(T)/T)$ or $O(1/T)$ for a strongly convex objective.

**Batch type:** linear convergence.
Reminder of online stochastic optimization

\[ g_t \in \partial \ell_t(x_t), \quad \bar{g}_t = \frac{1}{t} \sum_{\tau=1}^{t} g_\tau \quad (\ell_t(x) = \ell(z_t, x)) \]

**OPG (Online Proximal Gradient Descent)**

\[ x_{t+1} = \arg \min_x \left\{ \langle g_t, x \rangle + \tilde{\psi}(x) + \frac{1}{2\eta_t} \|x - x_t\|^2 \right\} \]

**RDA (Regularized Dual Averaging; Xiao (2009), Nesterov (2009))**

\[ x_{t+1} = \arg \min_x \left\{ \langle \bar{g}_t, x \rangle + \tilde{\psi}(x) + \frac{1}{2\eta_t} \|x\|^2 \right\} \]

These update rule is computed by a proximal mapping associated with \( \tilde{\psi} \).

- Efficient for a simple regularization such as \( \ell_1 \) regularization.
- How about structured regularization? \( \rightarrow \) ADMM.
OPG-ADMM

Ordinary OPG: \( x_{t+1} = \arg\min_x \left\{ \langle g_t, x \rangle + \tilde{\psi}(x) + \frac{1}{2 \eta_t} \| x - x_t \|^2 \right\} \).

\[
\begin{align*}
x_{t+1} &= \arg\min_{x \in X} \left\{ g_t^\top x - \lambda_t^\top (B^\top x - y_t) + \frac{\rho}{2} \| B^\top x - y_t \|^2 + \frac{1}{2 \eta_t} \| x - x_t \|^2_{G_t} \right\}, \\
y_{t+1} &= \arg\min_{y \in Y} \left\{ \psi(y) - \lambda_t^\top (B^\top x_{t+1} - y) + \frac{\rho}{2} \| B^\top x_{t+1} - y \|^2 \right\}
\end{align*}
\]

\[\lambda_{t+1} = \lambda_t - \rho (B^\top x_{t+1} - y_{t+1}).\]

- The update rule of \( y_{t+1} \) and \( \lambda_{t+1} \) are same as the ordinary ADMM.
- \( \text{prox}(\cdot | \psi) \) is usually analytically obtained.
- \( G_t \) is any positive definite matrix.
The update rule of $y_{t+1}$ and $\lambda_{t+1}$ are same as the ordinary ADMM.

$
prox(\cdot | \psi)$ is usually analytically obtained.

$G_t$ is any positive definite matrix.
RDA-ADMM

Ordinary RDA: \( w_{t+1} = \arg \min_w \left\{ \langle \tilde{g}_t, w \rangle + \tilde{\psi}(w) + \frac{1}{2\eta_t} \|w\|^2 \right\} \)

RDA-ADMM

Let \( \tilde{x}_t = \frac{1}{t} \sum_{\tau=1}^{t} x_\tau \), \( \tilde{\lambda}_t = \frac{1}{t} \sum_{\tau=1}^{t} \lambda_\tau \), \( \tilde{y}_t = \frac{1}{t} \sum_{\tau=1}^{t} y_\tau \), \( \tilde{g}_t = \frac{1}{t} \sum_{\tau=1}^{t} g_\tau \).

\[
x_{t+1} = \arg\min_{x \in X} \left\{ \tilde{g}_t^\top x - (B\tilde{\lambda}_t)^\top x + \frac{\rho}{2t} \|B^\top x\|^2 \right. \\
+ \rho (B^\top \tilde{x}_t - \tilde{y}_t)^\top B^\top x + \frac{1}{2\eta_t} \|x\|_2^2 \left\} ,
\]

\[
y_{t+1} = \text{prox}(B^\top x_{t+1} - \lambda_t / \rho |\psi|),
\]

\[
\lambda_{t+1} = \lambda_t - \rho (B^\top x_{t+1} - y_{t+1}).
\]

The update rule of \( y_{t+1} \) and \( \lambda_{t+1} \) are same as the ordinary ADMM.
Simplified version

Letting $G_t$ be a specific form, the update rule is much simplified. 
($G_t = \gamma I - \frac{\rho}{t} \eta_t BB^\top$ for RDA-ADMM, and $G_t = \gamma I - \rho \eta_t BB^\top$ for OPG-ADMM)

Online stochastic ADMM

1. **Update of $x$**

   (OPG-ADMM) \( x_{t+1} = \Pi_{\mathcal{X}} \left[ -\eta_t \left\{ g_t - B(\lambda_t - \rho B^\top x_t + \rho y_t) \right\} + x_t \right] \),

   (RDA-ADMM) \( x_{t+1} = \Pi_{\mathcal{X}} \left[ -\eta_t \left\{ \bar{g}_t - B(\tilde{\lambda}_t - \rho B^\top \bar{x}_t + \rho \bar{y}_t) \right\} \right] \).

2. **Update of $y$**

   \( y_{t+1} = \text{prox}(B^\top x_{t+1} - \lambda_t / \rho | \psi) \).

3. **Update of $\lambda$**

   \( \lambda_{t+1} = \lambda_t - \rho (B^\top x_{t+1} - y_{t+1}) \).

- Fast computation.
- Easy implementation.
Convergence analysis

We bound the expected risk:

- Expected risk

\[ P(x) = \mathbb{E}_Z[\mathcal{L}(Z, x)] + \tilde{\psi}(x). \]

Assumptions:

(A1) \( \exists G \text{ s.t. } \forall g \in \partial_x \ell(z, x) \text{ satisfies } \|g\| \leq G \text{ for all } z, x. \)

(A2) \( \exists L \text{ s.t. } \forall g \in \partial \psi(y) \text{ satisfies } \|g\| \leq L \text{ for all } y. \)

(A3) \( \exists R \text{ s.t. } \forall x \in \mathcal{X} \text{ satisfies } \|x\| \leq R. \)
Convergence rate: bounded gradient

(A1) \ \exists G \text{ s.t. } \forall g \in \partial_x \ell(z, x) \text{ satisfies } \|g\| \leq G \text{ for all } z, x.

(A2) \ \exists L \text{ s.t. } \forall g \in \partial \psi(y) \text{ satisfies } \|g\| \leq L \text{ for all } y.

(A3) \ \exists R \text{ s.t. } \forall x \in \mathcal{X} \text{ satisfies } \|x\| \leq R.

Theorem (Convergence rate of RDA-ADMM)

Under (A1), (A2), (A3), we have

$$
E_{z_1:T-1} [P(\bar{x}_T) - P(x^*)] \leq \frac{1}{T} \sum_{t=2}^{T} \frac{\eta_{t-1}}{2(t-1)} G^2 + \frac{\gamma}{\eta_T} \|x^*\|^2 + \frac{K}{T}.
$$

Theorem (Convergence rate of OPG-ADMM)

Under (A1), (A2), (A3), we have

$$
E_{z_1:T-1} [P(\bar{x}_T) - P(x^*)] \leq \frac{1}{2T} \sum_{t=2}^{T} \max \left\{ \frac{\gamma}{\eta_t} - \frac{\gamma}{\eta_{t-1}}, 0 \right\} R^2 + \frac{1}{T} \sum_{t=1}^{T} \eta_t G^2 + \frac{K}{T}.
$$

Both methods have convergence rate $O\left(\frac{1}{\sqrt{T}}\right)$ by letting $\eta_t = \eta_0 \sqrt{t}$ for RDA-ADMM and $\eta_t = \eta_0 / \sqrt{t}$ for OPG-ADMM.
Convergence rate: strongly convex

(A4) There exist $\sigma_f, \sigma_\psi \geq 0$ and $P, Q \succeq O$ such that

$$
\mathbb{E}_Z[\ell(Z, x) + (x' - x)^\top \nabla_x \ell(Z, x)] + \frac{\sigma_f}{2} \|x - x'\|^2_P \leq \mathbb{E}_Z[\ell(Z, x')],
$$

$$
\psi(y) + (y' - y)^\top \nabla \psi(y) + \frac{\sigma_\psi}{2} \|y - y'\|^2_Q,
$$

for all $x, x' \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$, and $\exists \sigma > 0$ satisfying

$$
\sigma I \preceq \sigma_f P + \frac{\rho \sigma_\psi}{2\rho + \sigma_\psi} Q.
$$

The update rule of RDA-ADMM is modified as

$$
x_{t+1} = \arg\min_{x \in \mathcal{X}} \left\{ \tilde{g}_t^\top x - \tilde{\lambda}_t^T B^T x + \frac{\rho}{2t} \|B^T x\|^2 + \rho (B^T \tilde{x}_t - \tilde{y}_t)^T B^T x + \frac{1}{2\eta_t} \|x\|^2_{G_t} \right. \left. + \frac{\sigma}{2} \|x - \tilde{x}_t\|^2 \right\}.
$$
Convergence analysis: strongly convex

Theorem (Convergence rate of RDA-ADMM)
Under (A1), (A2), (A3), (A4), we have
\[
\mathbb{E}_{z_1:T-1} [P(\tilde{x}_T) - P(x^*)] \leq \frac{1}{2T} \sum_{t=2}^{T} \frac{1}{\eta_t} + t\sigma \ G^2 + \frac{\gamma}{\eta_T} \|x^*\|^2 + \frac{K}{T}.
\]

Theorem (Convergence rate of OPG-ADMM)
Under (A1), (A2), (A3), (A4), we have
\[
\mathbb{E}_{z_1:T-1} [P(\tilde{x}_T) - P(x^*)] \leq \frac{1}{2T} \sum_{t=2}^{T} \max \left\{ \frac{\gamma}{\eta_t} - \frac{\gamma}{\eta_{t-1}} - \sigma, 0 \right\} R^2 \\
+ \frac{1}{T} \sum_{t=1}^{T} \frac{\eta_t}{2} G^2 + \frac{K}{T}.
\]

Both methods have convergence rate \( O\left(\frac{\log(T)}{T}\right) \) by letting \( \eta_t = \eta_0 t \) for RDA-ADMM and \( \eta_t = \eta_0 / t \) for OPG-ADMM.
This can be improved to \( O\left(1/T\right) \) by weighted averaging (Azadi and Sra, 2014).
Numerical experiments

**Figure:** Artificial data: 1024 dim, 512 sample, Overlapped group lasso.

**Figure:** Real data (Adult, a9a): 15,252 dim, 32,561 sample, Overlapped group lasso + $\ell_1$ reg.
Related methods

- $O(n/T)$ (improved from $O(1/\sqrt{T})$) convergence in a batch setting: Zhong and Kwok (2014)
- Acceleration of stochastic ADMM: Azadi and Sra (2014)
- Parallel computing with stochastic ADMM: Wang et al. (2014)
Outline

1. Stochastic optimization for structured regularization
   - Structured regularization
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   - Stochastic ADMM for online data
   - Stochastic ADMM for batch data

2. Parallel and distributed optimization

3. Further interesting topics
Batch setting

In the batch setting, the data are fixed. We just minimize the objective function defined by

$$\frac{1}{n} \sum_{i=1}^{n} f_i(a_i^T x) + \psi(B^T x).$$

- ADMM version of SDCA
- Converges linearly:

$$T > (n + \gamma/\lambda) \log(1/\epsilon)$$

...to achieve $\epsilon$ accuracy for $\gamma$-smooth loss and $\lambda$-strongly convex regularization.
Dual problem

Let \( A = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{p \times n} \).

\[
\min_w \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i(a_i^\top w) + \psi(B^\top w) \right\} \quad \text{(P: Primal)}
\]

\[
= - \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^d} \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i^*(x_i) + \psi^* \left( \frac{y}{n} \right) \mid Ax + By = 0 \right\} \quad \text{(D: Dual)}
\]

Optimality condition:

\[
a_i^\top w^* \in \nabla f_i^*(x_i^*), \quad \frac{1}{n} y^* \in \nabla \psi(u) \mid_{u = B^\top w^*}, \quad Ax^* + By^* = 0.
\]

★ Each coordinate \( x_i \) corresponds to each observation \( a_i \).
Let the augmented Lagrangian be
\[ L(x, y, w) := \sum_{i=1}^{n} f_i^*(x_i) + n\psi^*(y/n) - \langle w, Ax + By \rangle + \frac{\rho}{2} \| Ax + By \|^2. \]

Basic algorithm

For each \( t = 1, 2, \ldots \)

Choose \( i \in \{1, \ldots, n\} \) uniformly at random, and update

\[ y^{(t)} \leftarrow \arg \min_y \left\{ L(x^{(t-1)}, y, w^{(t-1)}) + \frac{1}{2} \| y - y^{(t-1)} \|^2_Q \right\} \]

\[ x_i^{(t)} \leftarrow \arg \min_{x_i \in \mathbb{R}} \left\{ L([x_i; x_{\backslash i}^{(t-1)}], y^{(t)}, w^{(t-1)}) + \frac{1}{2} \| x_i - x_i^{(t-1)} \|^2_{G_{i,i}} \right\} \]

\[ w^{(t)} \leftarrow w^{(t-1)} - \xi \rho \{ n(Ax^{(t)} + By^{(t)}) - (n - 1)(Ax^{(t-1)} + By^{(t-1)}) \}. \]

\( Q, G_{i,i} \) are positive definite matrices that satisfy some condition.

- Only \( i \)-th coordinate \( x_i \) is updated.
- The update of the multiplier \( w \) should be modified.
Let the augmented Lagrangian be
\[ \mathcal{L}(x, y, w) := \sum_{i=1}^{n} f_i^*(x_i) + \psi^*(y/n) - \langle w, Ax + By \rangle + \frac{\rho}{2} \|Ax + By\|^2. \]

Split the index set \( \{1, \ldots, n\} \) into \( K \) groups \((l_1, l_2, \ldots, l_K)\).

**Block coordinate SDCA-ADMM**

For each \( t = 1, 2, \ldots \)

Choose \( k \in \{1, \ldots, K\} \) uniformly at random, and set \( l = l_k \),

\[ y^{(t)} \leftarrow \arg \min_y \left\{ \mathcal{L}(x^{(t-1)}, y, w^{(t-1)}) + \frac{1}{2} \|y - y^{(t-1)}\|^2_Q \right\} \]

\[ x_l^{(t)} \leftarrow \arg \min_{x_l \in \mathbb{R}^{|l|}} \left\{ \mathcal{L}([x_l; x_{\backslash l}^{(t-1)}], y^{(t)}, w^{(t-1)}) + \frac{1}{2} \|x_l - x_l^{(t-1)}\|^2_{G_l,l} \right\} \]

\[ w^{(t)} \leftarrow w^{(t-1)} - \xi \rho \left\{ n(Ax^{(t)} + By^{(t)}) - (n - n/K)(Ax^{(t-1)} + By^{(t-1)}) \right\}. \]

\( Q, G_{l,l} \) are positive definite matrices that satisfy some condition.
Simplified algorithm

Setting

\[ Q = \rho(\eta_B I_d - B^\top B), \quad G_{I, I} = \rho(\eta_{Z, I} I_{|I|} - Z_I^\top Z_I), \]

then, by the relation \( \prox(q|\psi) + \prox(q|\psi^*) = q \), we obtain the following update rule:

**Simplified algorithm**

- For \( q^{(t)} = y^{(t-1)} + \frac{B^\top}{\rho \eta_B} \{ w^{(t-1)} - \rho(Zx^{(t-1)} + By^{(t-1)}) \} \), let

  \[ y^{(t)} \leftarrow q^{(t)} - \prox(q^{(t)}|m\psi(\rho \eta_B \cdot )/(\rho \eta_B)), \]

- For \( p_{I}^{(t)} = x_{I}^{(t-1)} + \frac{Z_I^\top}{\rho \eta_{Z, I}} \{ w^{(t-1)} - \rho(Zx^{(t-1)} + By^{(t)}) \} \), let

  \[ x_{I}^{(t)} \leftarrow \prox(p_{I}^{(t)}|f_i^*/(\rho \eta_{Z, I})) \quad (\forall i \in I). \]

\[ \star \] The update of \( x \) can be parallelized.
Convergence Analysis

$x^*$: the optimal variable of $x$
$\mathcal{Y}^*$: the set of optimal variables (not necessarily unique)
$w^*$: the optimal variable of $w$

Assumption:

- There exits $\nu > 0$ such that, $\forall x_i \in \mathbb{R}$,
  $$f_i^*(x_i) - f_i^*(x_i^*) \geq \langle \nabla f_i^*(x_i^*), x_i - x_i^* \rangle + \frac{\|x_i - x_i^*\|^2}{2\gamma}.$$  

- $\exists h, \nu_\psi > 0$ such that, for all $y, u$, there exists $\hat{y}^* \in \mathcal{Y}^*$ such that
  $$\psi^*(y/n) - \psi^*(\hat{y}^*/n) \geq \langle B^\top w^*, y/n - \hat{y}^*/n \rangle + \frac{\nu_\psi}{2} \|P_{\text{Ker}(B)}(y/n - \hat{y}^*/n)\|^2,$$
  $$\psi(u) - \psi(B^\top w^*) \geq \langle y^*/n, u - B^\top w^* \rangle + \frac{\lambda}{2} \|u - B^\top w^*\|^2.$$  

- $B^\top$ is injective.
\[ F_D(x, y) := \frac{1}{n} \sum_{i=1}^{n} f_i^*(x_i) + \psi^*(\frac{y}{n}) - \langle w^*, A \frac{x}{n} - B \frac{y}{n} \rangle. \]
\[ R_D(x, y, w) := F_D(x, y) - F_D(x^*, y^*) + \frac{1}{2n^2\xi\rho} \| w - w^* \|^2 + \frac{\rho(1-\xi)}{2n} \| Ax + By \|^2 + \frac{1}{2n} \| x - x^* \|^2 _{1p/\gamma + H} + \frac{1}{2nK} \| y - y^* \|^2 _Q. \]

**Theorem (Linear convergence of SDCA-ADMM)**

Let \( H \) be a matrix such that \( H_{i,i} = \rho A_i^\top A_i + G_{i,i} \) for all \( i \in \{1, \ldots, I_K\} \),

\[
\mu = \min \left\{ \frac{1}{4(1 + \gamma \sigma_{\text{max}}(H))}, \frac{\lambda \rho \sigma_{\text{min}}(B^\top B)}{2 \max\{1/n, 4\lambda \rho, 4\lambda \sigma_{\text{max}}(Q)\}}, \frac{K V_{\psi}/n}{4 \sigma_{\text{max}}(Q)}, \frac{K \sigma_{\text{min}}(BB^\top)}{4 \sigma_{\text{max}}(Q)(\rho \gamma \sigma_{\text{max}}(A^\top A) + 4)} \right\},
\]

\( \xi = \frac{1}{4n} \), and \( C_1 = R_D(x(0), y(0), w(0)) \), then we have that

- (dual residual) \( \mathbb{E}[R_D(x(t), y(t), w(t))] \leq \left(1 - \frac{\mu}{K}\right)^t C_1 \),
- (primal variable) \( \mathbb{E}[\| w(t) - w^* \|^2] \leq \frac{n \rho}{2} \left(1 - \frac{\mu}{K}\right)^t C_1. \)

For \( t \geq C' \frac{K}{\mu} \log \left( \frac{C'' n}{\epsilon} \right) \), we have that \( \mathbb{E}[\| w(t) - w^* \|^2] \leq \epsilon. \)
Convergence Analysis

Assumption:

- \( f_i \) is \( \gamma \)-smooth.
- \( \tilde{\psi} \) is \( \lambda \)-strongly convex.
- Other technical conditions.

With a setting \( \rho = \min\{1, 1/\gamma\} \) and \( K = n \),

\[
t \geq C \left( n + \frac{\gamma}{\lambda} \right) \log \left( \frac{C'}{\epsilon} \right)
\]

gives \( E[\|w(t) - w^*\|^2] \leq \epsilon \) (\( \frac{\gamma}{\lambda} \) is like the condition number).
The rate is as good as the ordinary SDCA.

Non stochastic method (e.g. ADMM in dual):

\[
t \geq C n \frac{\gamma}{\lambda} \log \left( \frac{C'}{\epsilon} \right) .
\]
Numerical Experiments: Loss function

Binary classification.

Smoothed hinge loss: \[ f_i(u) = \begin{cases} 
0, & (y_i u \geq 1), \\
\frac{1}{2} - y_i u, & (y_i u < 0), \\
\frac{1}{2}(1 - y_i u)^2, & \text{(otherwise)}.
\end{cases} \]

\[ \text{proximal mapping is analytically obtained.} \]

\[ \text{prox}(u|f_i^*/C) = \begin{cases} 
\frac{Cu - y_i}{1+C}, & (-1 \leq \frac{Cu_i - 1}{1+C} \leq 0), \\
-y_i, & (-1 > \frac{Cu_i - 1}{1+C}), \\
0, & \text{(otherwise)}.
\end{cases} \]
Numerical Experiments (Artificial data): Setting

Artificial data: Overlapped group regularization:

\[ \tilde{\psi}(X) = C \left( \sum_{i=1}^{32} \|X_{:,i}\| + \sum_{j=1}^{32} \|X_{j,:}\| + 0.01 \times \sum_{i,j} X_{i,j}^2 / 2 \right), \]

\( X \in \mathbb{R}^{32 \times 32} \).
Numerical Experiments (Artificial data): Results

(a) Excess objective value

(b) Test loss

(c) Class. Error

Figure: Artificial data ($n=5,120$, $d=1024$). Overlapped group lasso. Mini-batch size $n/K = 50$. 

Numerical Experiments (Artificial data): Results

(a) Excess objective value  (b) Test loss  (c) Class. Error

Figure: Artificial data $(n=51,200, d=1024)$. Overlapped group lasso. Mini-batch size $n/K = 50$. 
Numerical Experiments (Real data): Setting

Real data: Graph regularization:

\[ \tilde{\psi}(w) = C_1 \sum_{i=1}^{p} |w_i| + C_2 \sum_{(i,j) \in E} |w_i - w_j| + 0.01 \times (C_1 \sum_{i=1}^{p} |w_i|^2 + C_2 \sum_{(i,j) \in E} |w_i - w_j|^2) \]

where \( E \) is the set of edges obtained from the similarity matrix.

The similarity graph is obtained by Graphical Lasso (Yuan and Lin, 2007, Banerjee et al., 2008).
Numerical Experiments (Real data): Results

(a) Objective function  
(b) Test loss  
(c) Class. Error

Figure: Real data (20 Newsgroups, $n=12,995$). Graph regularization. Mini-batch size is $n/K = 50$. 

Numerical Experiments (Real data): Results

(a) Objective function  
(b) Test loss  
(c) Class. Error

Figure: Real data (a9a, \(n=32,561\)). Graph regularization. Mini-batch size \(n/K = 50\).
Outline

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3. Further interesting topics
Distributed computing

Q: How to deal with huge data that cannot be treated in one computational node?
A: Distributed computing.

**Challenge** communication cost trade-off: communication inside node v.s. between nodes via network.

We briefly introduce two approaches

- Simple averaging of SGD: (Zinkevich et al., 2010, Zhang et al., 2013)
- Distributed dual coordinate descent (COCOA+): (Ma et al., 2015)
Simple averaging

Run independent SGDs by $K$ nodes.

Take the average of $K$ final solutions:

$$\hat{x}_K = \frac{1}{K} \sum_{k=1}^{K} x[k].$$

Just one synchronization, efficient communication cost. How about convergence?
Assumption:

- The loss function is sufficiently smooth (there exists second order derivative, and it is Lipschitz continuous and bounded).
- The expected loss is $\lambda$-strongly convex.
- Each node runs $T$ iterations of SGD.

**Theorem ((Zhang et al., 2013))**

*With an appropriate step size,*

\[
E[\|\hat{x}_K - x^*\|^2] \leq C \left( \frac{G^2}{KT\lambda^2} + \frac{1}{T^{3/2}} \right).
\]

- As $K$ is increased, the main term is linearly improved.
- However, too large $K$ is not effective. Actually, for $\lambda \in (0, 1/\sqrt{T})$, it is shown that (Shamir et al., 2014)

\[
E[\|\hat{x}_K - x^*\|^2] \geq \frac{C}{\lambda^2 T}.
\]
We divide the sample into $K$ groups $\{G_k\}_k$:

$$\{1, \ldots, n\} = \bigcup_{k=1}^{K} G_k, \ G_k \cap G_k' = \emptyset.$$ 

**Dual problem of RERM:**

$$D(y) = \frac{1}{n} \sum_{i=1}^{n} f_i^*(y_i) + \psi^* \left( -\frac{1}{n} Ay \right)$$

$$= \frac{1}{n} \sum_{k=1}^{K} \left( \sum_{i \in G_k} f_i^*(y_i) \right) + \psi^* \left( -\frac{1}{n} \sum_{k=1}^{K} A_{G_k} y_{G_k} \right)$$

Divided into $K$ groups

needs synchronization

where $A_{G_k} = [a_{i_1}, \ldots, a_{i_{|G_k|}}] \in \mathbb{R}^{p \times |G_k|}$ where $i_j \in G_k$ and $y_{G_k} = (y_i)_{i \in G_k}$.
Run small coord. desc. in parallel

\[
\sum_{k=1}^{K} A_{G_k} y_{G_k}
\]

Sum up the results (synchronization)

Run small coord. desc. in parallel
1. $f_i$ is $\gamma_f$-smooth.
2. $\psi$ is $\lambda$-strongly convex.
3. Each subproblem decreases the objective by a factor of $\Theta$.

Let the separability of the problems as

$$\sigma_{\min} := \max_{y \in \mathbb{R}^n} \frac{||Ay||^2}{\sum_{k=1}^{K} ||A_{G_k}y_{G_k}||^2}, \quad \sigma_{\max} := \max_k \max_{y_{G_k} \in \mathbb{R}^{\mathcal{G}_k}} \frac{||A_{G_k}y_{G_k}||^2}{||y_{G_k}||^2}.$$  

### Theorem

**Under an appropriate setting of parameters, after $T$ iterations with**

$$T \geq C \frac{\sigma_{\min} \sigma_{\max} \gamma_f/(\lambda n) + 1}{(1-\Theta)} \log(1/\epsilon),$$

**it holds that** $E[D(y^{(T)}) - D(y^*)] \leq \epsilon$. **Furthermore, after $T$ iterations with**

$$T \geq C \frac{\sigma_{\min} \sigma_{\max} \gamma_f/\lambda + 1}{(1-\Theta)} \log(\frac{\sigma_{\min} \sigma_{\max} \gamma_f/\lambda + 1}{(1-\Theta)} / \epsilon),$$

**it holds that** $E[P(w^{(t)}) - D(y^{(t)})] \leq \epsilon$.  

It is shown that $\sigma_{\text{min}} \leq K$ and $\sigma_{\text{max}} \leq n/K$. Then

$$T \geq C \frac{\gamma_f/\lambda + 1}{(1 - \Theta)} \log(1/\epsilon)$$

achieves $\epsilon$ accuracy. This is equivalent to iteration number of batch gradient methods on a strongly convex function.

Typically $\Theta = \left(1 - \frac{1}{n/K + \gamma_f/\lambda}\right)^t$ where $t$ is the number of inner iterations.

Total computational time (worst case):

$$t(1 + \gamma_f/\lambda) \log(1/\epsilon) = \left(\frac{n}{K} + \frac{\gamma_f}{\lambda}\right) \left(1 + \frac{\gamma_f}{\lambda}\right) \log(1/\epsilon).$$

Huge learning problems can be optimized on a distributed system with linear convergence rate.
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Stochastic primal dual coordinate method

(Zhang and Lin, 2015)

\[
\min_x \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i(a_i^T x) + \psi(x) \right\}
\]

\( x \) and \( y_i \) (\( i \) is chosen randomly) are updated alternatively.

Iteration complexity:

\[
T \geq \left( n + \sqrt{\gamma/\lambda} \right) \log(1/\epsilon)
\]
Stochastic primal dual coordinate method

(Zhang and Lin, 2015)

\[
\min_x \left\{ \frac{1}{n} \sum_{i=1}^{n} f_i(a_i^\top x) + \psi(x) \right\}
\]

\[
\sup_{y_i} \{ \langle x, a_i y_i \rangle - f_i^*(y_i) \}
\]

\[
= \min_x \max_y \left\{ \frac{1}{n} \sum_{i=1}^{n} (\langle x, a_i y_i \rangle - f_i^*(y_i)) + \psi(x) \right\}
\]

\text{x and } y_i \ (i \text{ is chosen randomly}) \text{ are updated alternatively.}

Iteration complexity:

\[
T \geq \left( n + \sqrt{\gamma/\lambda} \right) \log(1/\epsilon)
\]
Multi-armed bandit

Maximize the sum of rewards earned through a sequence of plays.

- Formulated by Robbins (1952).
- Optimal strategy: Lai and Robbins (1985)
- Thompson sampling (Bayesian strategy): Thompson (1933)

Continuous version of bandit: **Bayesian optimization** (Močkus, 1975, Mockus and Mockus, 1991, Srinivas et al., 2012, Snoek et al., 2012)

- Gaussian process regression to search the peak of a function.
- Practically useful for hyper-parameter tuning of deep learning.
Goal: Efficient sampling from the posterior distribution.

Randomly choose small mini-batch $I_t \subseteq \{1, \ldots, n\}$ and update the parameter $\theta_t$:

$$
\theta_t = \theta_{t-1} + \eta_t \left[ \nabla_\theta \log \pi(\theta) + \frac{n}{|I_t|} \sum_{i \in I_t} \nabla_\theta \log (p(x_i|\theta)) \right] + \epsilon_t \xi
$$

where $\xi \sim N(0, \eta_t I)$ and

$$
\sum_{t=1}^{\infty} \epsilon_t = \infty, \quad \sum_{t=1}^{\infty} \epsilon_t^2 < \infty.
$$

Related topic: Stochastic variational inference (Hoffman et al., 2013).

- Bayesian inference for large datasets.
- Practically very useful.
- Supported by some theoretical backgrounds.
Connection to learning theory

\[
\min_x \left\{ \frac{1}{t} \sum_{\tau=1}^{t} \ell(z_\tau, x) + \lambda_t \|x\|_1 \right\}
\]

- Solve regularized learning problem in online setting (data \(z_t\) comes one after another).
- The regularization parameter \(\lambda_t\) should go to zero as \(t \to \infty\).

**Question:** Can we achieve statistically optimal estimation?  
→ **Yes.**

- **RADAR** (\(L_1\)-regularization): Agarwal et al. (2012)
- **REASON** (Stochastic ADMM): Sedghi et al. (2014)

\[
\|x_t - x^*\|^2 \leq \frac{s \log(p)}{T}
\]

for \(s\)-sparse truth \(x^*\).

**Simultaneous discussions of optimization and statistics.**
Summary of part III

- Stochastic ADMM for structured sparsity
  - Online proximal gradient method, regularized dual averaging method for ADMM (online)
  - Stochastic dual coordinate descent for ADMM (batch)
  - A similar convergence result to the normal ones

- Distributed stochastic optimization
  - Simple averaging
  - Distributed SDCA (COCOA+)


Y. Kawahara, K. Nagano, K. Tsuda, and J. A. Bilmes. Submodularity cuts


